

LINEAR INDEPENDENCE OF KNOTS ARISING FROM ITERATED INFECTION WITHOUT THE USE OF TRISTRAM LEVINE SIGNATURE.

CHRISTOPHER DAVIS

ABSTRACT. We give an explicit construction of linearly independent families of knots arbitrarily deep in the (n) -solvable filtration of the knot concordance group using the ρ^1 -invariant defined in [12]. A difference between previous constructions of infinite rank subgroups in the concordance group and ours is that the deepest infecting knots in the construction we present are allowed to have vanishing Tristram-Levine signatures.

1. INTRODUCTION

A knot K is an isotopy class of oriented locally flat embeddings of the circle S^1 into the 3-sphere S^3 . A pair of knots K and J are called **topologically concordant** if there is a locally flat embedding of the annulus $S^1 \times [0, 1]$ into $S^3 \times [0, 1]$ mapping $S^1 \times \{1\}$ to a representative of K in $S^3 \times \{1\}$ and $S^1 \times \{0\}$ to a representative of J in $S^3 \times \{0\}$. A knot is called **slice** if it is concordant to the unknot or equivalently if it is the boundary of a locally flat embedding of the 2-ball B^2 into the 4-ball B^4 . The set of all knots modulo concordance under the operation of connected sum is a group called the **knot concordance group** and is denoted by \mathcal{C} .

In [10], Cochran, Orr and Teichner define the solvable filtration of \mathcal{C} :

$$\dots \mathcal{F}_{n.5} \subseteq \mathcal{F}_n \subseteq \dots \subseteq \mathcal{F}_{1.5} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}.$$

For k a half integer, the elements in \mathcal{F}_k are called **(k) -solvable**. They show that \mathcal{F}_0 is the set of Arf-invariant zero knots, $\mathcal{F}_{0.5}$ is the set of algebraically slice knots and that Casson-Gordon invariants vanish on $\mathcal{F}_{1.5}$. In [10, Section 6] Cochran-Orr-Teichner show that $\mathcal{F}_2/\mathcal{F}_{2.5}$ is infinite rank by studying a satellite operation (called infection in [6, Section 8]). The quotient groups $\mathcal{F}_n/\mathcal{F}_{n.5}$ have been an active place of research ever since. In [9] Cochran, Harvey and Leidy begin with knots for which the integrals of the Tristram-Levine signature functions are linearly independent over \mathbb{Q} and use an iterated infection procedure to produce an infinite rank free Abelian subgroup of $\mathcal{F}_n/\mathcal{F}_{n.5}$.

Date: September 20, 2011.

2000 Mathematics Subject Classification. 46L55.

In [3] Cha constructs another infinite rank subgroup of $\mathcal{F}_n/\mathcal{F}_{n.5}$ starting instead with knots whose Tristram-Levine signatures evaluate to sufficiently large values at particular finite sets.

We present a variation on this idea, performing iterated infections to produce linearly independent sets deep in the solvable filtration. A novel aspect of the construction we present is that the deepest infecting knots are allowed have vanishing Tristram-Levine signature. A concrete advantage of this construction over previous ones is that its conditions may be directly verified, providing explicit infinite linearly independent sets rather than generating sets for infinite rank subgroups.

Given any closed oriented 3-manifold M and a homomorphism $\phi : \pi_1(M) \rightarrow \Gamma$, the von Neumann ρ -invariant, $\rho(M, \phi) \in \mathbb{R}$, is defined. It is an invariant of orientation preserving homeomorphism of the pair (M, ϕ) . Restricting this invariant to the zero surgery of knots and links gives rise to an isotopy invariant. We provide a brief overview of ρ -invariants in Section 2.

In [12] the author defines a particular ρ -invariant, ρ^1 , shows that in a restricted setting it provides a concordance obstruction and computes it for an infinite family of twist knots of order 2 in the algebraic concordance group. In this paper we bring this invariant to bear on an iterated infection procedure in order to produce examples whose deepest infecting knots have vanishing Tristram-Levine signature.

For an overview of infection, see [6, section 8]. We denote the infection of the base knot R along the infecting curve η in $S^3 - R$ by the infecting knot J as either $R_\eta(J)$ or $R(\eta, J)$ depending on notational convenience.

Recall that for a knot K , the **rational Alexander module** of K , $A_0(K)$, is given by the first homology with coefficients in \mathbb{Q} of the infinite cyclic cover of the exterior of K or equivalently of the zero surgery of K , $M(K)$. The rational Alexander module of K is a module over the ring of Laurent polynomials, $\mathbb{Q}[t^{\pm 1}]$. With respect to the involution on $\mathbb{Q}[t^{\pm 1}]$ given by $\overline{p(t)} = p(t^{-1})$, there is a sesquilinear form

$$Bl : A_0(K) \times A_0(K) \rightarrow \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]},$$

called the **Blanchfield form**. A submodule $P \subseteq A_0(K)$ is called **isotropic** if $Bl(x, y)$ vanishes for all $x, y \in P$.

We now give definitions of the concepts needed in the statement of Theorem 1.1, the main theorem of this paper. They will be recalled when needed.

Two polynomials $p(t), q(t)$ are called **strongly coprime** if $p(t^k)$ and $q(t^l)$ have no common roots in \mathbb{C} for every choice of nonzero integers k and l . A pair (R, η) with R a knot and η a curve in its complement is called **doubly anisotropic** if η represents an element of $A_0(R)$ for which there does not exist any α and β in $A_0(R)$ with $\eta = \alpha + \beta$ and $Bl(\alpha, \alpha) = Bl(\beta, \beta) = 0$.

Theorem 1.1. *Let $\{K_i\}$ be a possibly infinite set of knots:*

- (1) *whose Alexander polynomials are strongly coprime,*

- (2) whose Tristram-Levine signatures have vanishing integrals,
- (3) whose prime factors have square-free Alexander polynomials and
- (4) whose ρ^1 -invariants do not vanish, that is $\rho^1(K_i) \neq 0$.

For $i = 1, 2, \dots$ and $j = 1, 2, \dots, n$ let $R_{i,j}$ be a slice knot and $\eta_{i,j}$ be an unknotted curve in the complement of $R_{i,j}$ such that the pair $(R_{i,j}, \eta_{i,j})$ is doubly anisotropic. Let $K_i^0 = K_i$ and $K_i^j = R_{i,j}(\eta_{i,j}, K_i^{j-1})$.

Then $\{K_i^n\}_{i=1}^\infty$ is linearly independent in \mathcal{C} modulo $(n+1.5)$ solvable knots.

A noteworthy difference between this and previous results producing infinite rank subgroups via iterated infection is the condition on the Tristram-Levine signature. Previous constructions assume that the deepest infecting knots, K_i , be complicated in some sense. In [9, Theorem 7.5] their integrals are required to be rationally linearly independent. In [3, Lemma 4.11 and Proposition 4.12] they are required to take large values at a specific set of points. By contrast, Theorem 1.1 is designed to apply even when Levine-Tristram signature functions vanish.

Another key difference between this result and such previous results is the ease of verifying that the assumptions of the theorem are satisfied. Specifically, notice that in Theorem 1.1 there is no assumption on the ρ -invariants of the slice knots $R_{i,j}$. The techniques of [9] require a condition on first order von Neumann ρ -invariants of the knots along which infection is performed. Without a means of computation, they cannot verify that any fixed knot satisfies this condition. The techniques of [3] require that the Tristram-Levine signature of the infecting knots exceed all of the von Neumann ρ -invariants of the knots along which infection is done. Without any means of getting concrete bounds, these techniques will not give any explicit linearly independent sets.

In Section 3, as an application of Theorem 1.1 we generate the following family of linearly independent knots deep in the filtration using the base knot R , infecting curve η (see Figure 1) and deepest infecting knot given by twist knots T_n of finite algebraic order (see Figure 2).

Theorem 3.3. *For the slice knot R and infecting curve η , if T_n is the n -twist knot then*

$$\{(R_\eta)^m(T_n) = (R_\eta \circ \dots \circ R_\eta)(T_n) | n = -x^2 - x - 1, x \geq 2\}$$

is linearly independent in $\mathcal{F}_{m-.5}/\mathcal{F}_{m+1.5}$, where $\mathcal{F}_{-.5}$ is taken to be the whole concordance group.

This family of knots appears to be the first linearly independent set deep in the solvable filtration of \mathcal{C} constructed by an iterated infection procedure with deepest infecting knots whose Tristram-Levine signature functions vanish.

1.1. Outline of the paper. In Section 2 we provide what in this paper is taken as the definition of the von Neumann ρ -invariant, as well as some

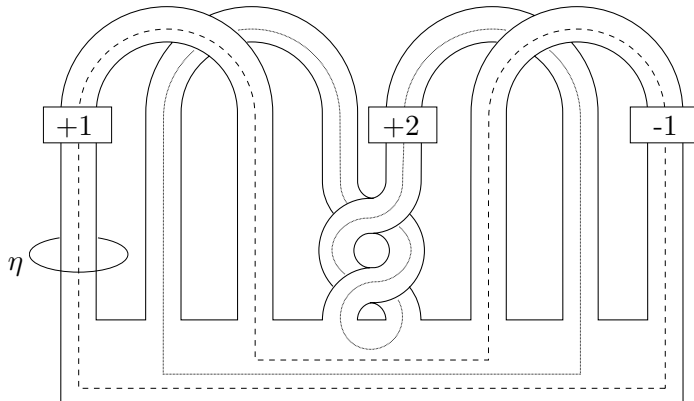


FIGURE 1. A slice knot R with a doubly anisotropic curve, η . The depicted derivative is the unlink.

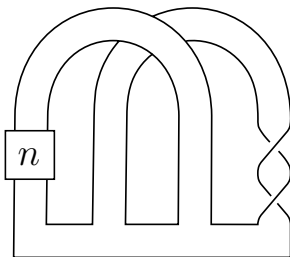


FIGURE 2. T_n , the n -twist knot.

properties of the L^2 -signature. We go on to provide definitions of the ρ^0 and ρ^1 -invariants which are employed in this paper.

In Section 3, we find a family of knots with nonzero ρ^1 -invariant whose Alexander polynomials are strongly coprime and square-free, as well as a set of slice knots whose rational Alexander modules have doubly anisotropic elements, that is, a set of knots satisfying the conditions of Theorem 1.1. This provides an explicit linearly independent set of knots sitting arbitrarily deep in the solvable filtration of the concordance group. The remaining sections are devoted to the development of the machinery used in the proof of Theorem 1.1

In Section 4 we discuss a localization of the Alexander module, $\widetilde{A}_0^p(K)$. In order to capture information involving this localization we define a new class of von Neumann ρ -invariant. It enjoys additivity properties over connected sum and infection and in some cases agrees with ρ^1 .

In Section 5 we study the Blanchfield linking form on $\widetilde{A}_0^p(K)$ and find sufficient conditions for this localized Blanchfield form to have no nontrivial isotropic submodules. The significance of this result appears in Section 6

in which we show that isotropic submodules of this localization are well behaved with respect to the operation of extension of scalars.

Finally, in Section 7 we give the proof of Theorem 1.1.

2. BACKGROUND: VON NEUMANN ρ -INVARIANTS AND L^2 SIGNATURES

In this section we state the properties of von Neumann ρ -invariants and L^2 signatures needed in this paper.

In [15, Section 3] the following property is proven of the von Neumann ρ -invariant. It serves here as the definition.

Definition 2.1. Consider oriented 3-manifolds M_1, \dots, M_n , with homomorphisms $\phi_i : \pi_1(M_i) \rightarrow \Gamma_i$. Suppose that $M_1 \sqcup M_2 \sqcup \dots \sqcup M_n$ is the oriented boundary of a compact oriented 4-manifold W and $\psi : \pi_1(W) \rightarrow \Lambda$ is a homomorphism such that, for each i , there is a monomorphism $\alpha_i : \Gamma_i \rightarrow \Lambda$ making the following diagram commute:

$$\begin{array}{ccc} \pi_1(M_i) & \xrightarrow{\phi_i} & \Gamma_i \\ \downarrow i_* & & \downarrow \alpha_i \\ \pi_1(W) & \xrightarrow{\psi} & \Lambda \end{array}$$

Then $\sum_{i=1}^n \rho(M_i, \phi_i) = \sigma^{(2)}(W, \psi) - \sigma(W)$ where $\sigma(W)$ is the regular signature of W and $\sigma^{(2)}(W, \psi)$ is the L^2 signature of W twisted by the coefficient system ψ . The expression $\sigma^{(2)}(W, \psi) - \sigma(W)$ is called the signature defect of W with respect to ψ .

For a compact oriented 4-manifold W with coefficient system $\phi : \pi_1(W) \rightarrow \Gamma$, $\sigma^{(2)}(W, \Gamma) \in \mathbb{R}$ is defined. The properties of the L^2 signature which are used in this paper are Novikov additivity and a bound in terms of ranks of twisted second homology in the case that Γ is PTFA (Poly Torsion Free Abelian, see [10, Definition 2.1]).

Proposition (Novikov additivity, [10, Lemma 5.9 (3)]). *If compact oriented 4-manifolds W_1 and W_2 intersect in a single common boundary component, $W = W_1 \cup W_2$, and $i^1 : W_1 \rightarrow W$ and $i^2 : W_2 \rightarrow W$ are the inclusion maps, then for every homomorphism $\phi : \pi_1(W) \rightarrow \Gamma$, $\sigma^{(2)}(W, \phi) = \sigma^{(2)}(W_1, \phi \circ i_*^1) + \sigma^{(2)}(W_2, \phi \circ i_*^2)$.*

The second property is that when Γ is PTFA and more generally whenever $\mathbb{Q}[\Gamma]$ is an Ore domain,

$$(2.1) \quad \left| \sigma^{(2)}(W, \phi) \right| \leq \text{Rank}_{\mathbb{Q}[\Gamma]} \left(\frac{H_2(W; \mathbb{Q}[\Gamma])}{i_*[H_2(\partial W; \mathbb{Q}[\Gamma])]} \right)$$

where $i_* : H_2(\partial W; \mathbb{Q}[\Gamma]) \rightarrow H_2(W; \mathbb{Q}[\Gamma])$ is the inclusion induced map. This follows from the monotonicity of von Neumann dimension (see [19,

Lemma 1.4]) and the fact that the L^2 Betti number agrees with $\mathbb{Q}[\Gamma]$ rank when $\mathbb{Q}[\Gamma]$ is an Ore Domain (see [5, Lemma 2.4] or [14, Proposition 2.4]).

2.1. The ρ^0 and ρ^1 -invariants.

Definition 2.2. For a knot K , Let $\phi^0 : \pi_1(M(K)) \rightarrow \mathbb{Z}$ be the Abelianization map. Let $\rho^0(K) := \rho(M(K), \phi^0)$ be the corresponding ρ -invariant.

It is shown in [11, Proposition 5.1] that $\rho^0(K)$ is given by the integral of the Tristram-Levine signature function.

Recall that the rational derived series of a group G is given by setting $G_{\tau}^{(0)} = G$ and inductively defining $G_{\tau}^{(n+1)}$ to be the set of all $g \in G_{\tau}^{(n)}$ which are torsion in the Abelianization of $G_{\tau}^{(n)}$. This series is the most quickly descending series with the property that each of the successive quotients $G_{\tau}^{(n)} / G_{\tau}^{(n+1)}$ is TFA (Torsion Free Abelian).

Definition 2.3. For a knot K , let

$$\phi^1 : \pi_1(M(K)) \rightarrow \frac{\pi_1(M(K))}{\pi_1(M(K))_{\tau}^{(2)}}$$

be the quotient by the second term in the rational derived series. Let $\rho^1(K) := \rho(M(K), \phi^1)$ be the corresponding ρ -invariant.

In [12] the ρ^1 -invariant is shown to provide a sliceness obstruction and is used to find an infinite collection of twist knots of algebraic order 2 which is linearly independent in \mathcal{C} .

3. GENERATING EXPLICIT LINEARLY INDEPENDENT FAMILIES OF KNOTS ARBITRARILY DEEP IN THE SOLVABLE FILTRATION

In this section we verify the assumptions of Theorem 1.1 for an explicit set of knots.

We first address the restriction we put on the slice knots we infect and curves along which we infect them.

Definition 3.1. For a knot, R , an element of $A_0(R)$, η , is called **doubly anisotropic** if η cannot be written as a sum of isotropic elements, that is there do not exist any $\alpha, \beta \in A_0(R)$ with $Bl(\alpha, \alpha) = Bl(\beta, \beta) = 0$ and $\alpha + \beta = \eta$.

In this paper we concern ourselves with the operator $R_{\eta} : \mathcal{C} \rightarrow \mathcal{C}$ given by sending a knot J to the result of infection $R_{\eta}(J)$ where R is a slice knot and η is an unknotted curve representing a doubly anisotropic element of $A_0(R)$.

The following proposition serves to illustrate that there are many slice knots whose Alexander modules have doubly anisotropic elements.

Proposition 3.2. *Let K be a slice knot with cyclic Alexander module isomorphic to $\frac{\mathbb{Q}[t^{\pm 1}]}{(\delta(t)^2)}$ where δ is any symmetric polynomial with $\delta(1) = \pm 1$. If*

δ has a prime symmetric factor then η , the generator of $A_0(R)$, is doubly anisotropic.

Proof. Let q be the assumed prime symmetric factor of δ . Consider any element, $f\eta$, in the Alexander module ($f \in \mathbb{Q}[t^{\pm 1}]$). If $f\eta$ were an isotropic element then

$$0 = Bl(f\eta, f\eta) = \frac{f(t)f(t^{-1})r(t)}{\delta(t)^2} \in \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]}$$

where $(r, \delta) = 1$. Then q , being a factor of δ , must divide $f(t)f(t^{-1})r(t)$. Since q is prime and $(r, q) = 1$ q must divide $f(t)$ or $f(t^{-1})$. Since q is symmetric, it divides both. Thus, any isotropic element of the $A_0(R)$ (and so any sum of two isotropic elements) sits in the proper submodule $P = \langle q\eta \rangle$ and any element of $A_0(R) - P$ (for example, η , the generator of $A_0(R)$) is doubly anisotropic. \square

Cha ([4, Theorem 5.18]) shows that for every symmetric polynomial δ there is a ribbon knot with Alexander module of the form $\frac{\mathbb{Q}[t^{\pm 1}]}{(\delta(t)^2)}$, so that slice knots with doubly anisotropic curves abound. For the sake of concreteness, let R be the slice knot depicted in Figure 1 and Let η be the curve in $S^3 - R$, also depicted in Figure 1. The Alexander module of R is cyclic generated by η and has Alexander polynomial of the form $\delta(t)^2$ where $\delta(t) = t^2 - 3t + 1$ is a symmetric prime polynomial. By Proposition 3.2 (R, η) is doubly anisotropic.

What remains is to find infinitely many infecting knots whose Tristram-Levine signatures have vanishing integrals, whose ρ^1 -invariants are nonzero and whose Alexander polynomials are strongly coprime and square-free. For $n < 0$, the twist knot T_n (see Figure 2) is algebraically of finite order, so that the Tristram-Levine signature vanishes. It is shown in [12, Theorem 6.1] that for $n(x) = -x^2 - x - 1$, and $x \geq 2$ $\rho^1(T_{n(x)}) \neq 0$. Their Alexander polynomials are prime and so square-free.

Theorem 3.1 of [1] gives us that the strong coprimality condition is satisfied. A note on conventions, the knot which is called T_n in this paper is the reverse of the mirror image of the knot called T_{-n} in [1].

Theorem (Theorem 3.1, [1]). *For positive integers $m \neq n$ the Alexander polynomials $\Delta_{T_{-n}}$ and $\Delta_{T_{-m}}$ are strongly coprime.*

Thus, the slice knot R and infecting curve η , together with the deepest infecting knot $T_{n(x)}$ for $x \geq 2$ satisfy the assumptions of Theorem 1.1 and we see that

Theorem 3.3. *The set $\{(R_\eta)^m(T_n) | n = -x^2 - x - 1, x \geq 2\}$ is linearly independent in $\mathcal{F}_{m-0.5}/\mathcal{F}_{m+1.5}$, where $\mathcal{F}_{-0.5}$ is taken to be all of the concordance group.*

Starting with a different choice of R_η , we construct families of knots that are linearly independent in the concordance group but which many previous invariants fail to detect.

Theorem 3.4. *Let $p_m(t)$ denote the m th cyclotomic polynomial where m is divisible by three distinct prime numbers.*

Let R be a ribbon knot with cyclic Alexander module $A_0(R) \cong \frac{\mathbb{Q}[t^{\pm 1}]}{(p_m^2)}$. Let η be an unknotted curve representing a generator of $A_0(R)$. Let T_n be the n -twist knot.

Then $\{R_\eta(T_n) | n = -x^2 - x - 1 \text{ with } x \geq 2\}$ is linearly independent in $\mathcal{F}_{0.5}/\mathcal{F}_{2.5}$ (and so in \mathcal{C}); however, the Casson-Gordon sliceness obstruction of [2], the metabelian η -invariant obstruction of [13] and the (1.5)-solvability obstructions of [16], [10] and [3] all vanish for each element of this set.

Proof. The fact that this set is linearly independent is an immediate consequence of Theorem 1.1.

For every n the Alexander polynomial of $R_\eta(T_n)$ is the same as the Alexander polynomial of R , which is p_m^2 . By [18, Theorem 1.2], every prime power cyclic branched cover of $R_\eta(T_n)$ is a homology sphere. Thus, the metabelian η -invariants of [13] and the Casson-Gordon obstructions vanish.

In order to compute the obstructions of [10] (using the specialization of the obstruction to (1.5)-solvability of [8, Theorem 4.2]) and [16, Theorem 4.1], notice that there is only one Lagrangian submodule of $A_0(R_\eta(T_n)) \cong A_0(R)$, namely $P = \langle p_m \rangle$. We first compute the obstruction of [8]. By [7, Lemma 2.3]

$$\rho_P^1(R_\eta(T_n)) = \rho_P^1(R) + \rho^0(T_n).$$

As T_n is of finite algebraic order, $\rho^0(T_n) = 0$. Since R is slice and P is the only Lagrangian submodule of $A_0(R)$, [8, Theorem 4.2] implies that $\rho_P^1(R) = 0$. This completes the proof that this invariant vanishes.

In order to compute the obstruction of [16], let $x \in P$ and consider the map ϕ_x defined in that paper. By [7, Lemma 2.3],

$$\rho(R_\eta(T_n), \phi_x) = \rho(R, \phi_x) + \rho^0(T_n),$$

similarly to before, $\rho^0(T_n)$ vanishes and since R is slice and has only one Lagrangian submodule, P , [16, Theorem 1.1] implies that $\rho(R, \phi_x) = 0$ for all $x \in P$.

Finally, we check that the (1.5)-solvability obstruction of [3, Theorem 1.3] vanishes. Since R is slice, it follows that $\rho(M(R), \phi) = 0$ for some coefficient system $\phi : \pi_1(M(R)) \rightarrow \Gamma$ where $\Gamma^{(2)} = 0$, Γ is amenable and Γ is in Strebel's class $D(S)$ (See [3] for a definition) where S is either \mathbb{Q} or a finite cyclic group. By [7, Lemma 2.3], then

$$\rho(M(R_\eta(T_n)), \phi) = \rho(M(R), \phi) + \rho(M(T_n), \phi).$$

As we observed, $\rho(M(R), \phi) = 0$. By [3, Lemma 4.5], $\rho(M(T_n), \phi)$ is a sum or integral of the Tristram-Levine signature of T_n and so is zero. \square

We do not know if it is possible to use the (2.5)-solvability obstructions of [10] or [3] to show that these knots are not (2.5)-solvable.

4. STRONGLY LOCALIZED ρ -INVARIANTS

For a knot K and a polynomial p , the strongly localized ρ -invariant of K , $\widetilde{\rho}_p^1(K)$ is defined in terms of a localization of the Alexander module of K . We begin by describing this localization.

For polynomials $p, q \in \mathbb{Q}[t^{\pm 1}]$ we say that p and q are **strongly coprime** ([9, Definition 4.4]) denoted $\widetilde{(p, q)} = 1$ if there is no nonzero complex number z and integers m, n such that $p(z^m) = q(z^n) = 0$. Let

$$\widetilde{S}_p = \{q \in \mathbb{Q}[t^{\pm 1}] \mid \widetilde{(p, q)} = 1\}$$

be the multiplicative set consisting of polynomials strongly coprime to p .

Let

$$\widetilde{R}_p = \mathbb{Q}[t^{\pm 1}] \widetilde{S}_p^{-1} = \left\{ \frac{f}{g} \in \mathbb{Q}(t) \mid \widetilde{(g, p)} = 1 \right\}$$

be the **strong localization** of $\mathbb{Q}[t^{\pm 1}]$ at p . By [20, Theorem 10.30] \widetilde{R}_p is flat as a $\mathbb{Q}[t^{\pm 1}]$ module so that for a knot K , the first homology of $M(K)$ with coefficients in \widetilde{R}_p is given by $H_1(M(K); \widetilde{R}_p) \cong H_1(M(K); \mathbb{Q}[t^{\pm 1}]) \otimes_{\mathbb{Q}[t^{\pm 1}]} \widetilde{R}_p$. We call this module the **strongly localized Alexander module** and denote it by $\widetilde{A}_0^p(K)$.

Let $\pi_1(M(K))_{\widetilde{p}}^{(2)}$ be the kernel of the composition

$$(4.1) \quad \pi_1(M(K))^{(1)} \rightarrow \frac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))^{(2)}} \hookrightarrow A_0(K) \rightarrow \widetilde{A}_0^p(K).$$

Let $\widetilde{\phi}_1^p : \pi_1(M(K)) \rightarrow \frac{\pi_1(M(K))}{\pi_1(M(K))_{\widetilde{p}}^{(2)}}$ be the quotient map.

Definition 4.1. Let $\widetilde{\rho}_p^1(K) = \rho(M(K), \widetilde{\phi}_1^p)$ be the **strongly localized ρ -invariant of K at p** .

We will be flexible with notation. For any CW-complex X with infinite cyclic first homology generated by t we can similarly define $A_0(X)$, $\widetilde{A}_0^p(X)$ and $\pi_1(X)_{\widetilde{p}}^{(2)}$.

The strongly localized ρ -invariant shares many properties with the similarly defined localized ρ -invariant of [12]. The proofs of the following propositions are identical to proofs in [12] and so are omitted.

Proposition 4.2 ([12, Proposition 3.4]). *If Δ is the Alexander polynomial of a knot K , then*

- (1) $\widetilde{\rho}_\Delta^1(K) = \rho^1(K)$
- (2) *If p and Δ are strongly coprime, then $\widetilde{\rho}_p^1(K) = \rho^0(K)$.*

Proposition 4.3 ([12, Proposition 3.5]). *Let J and K be knots and η be an unknot in the complement of J such that J and η have zero linking number.*

- (1) $\widetilde{\rho}_p^1(J \# K) = \widetilde{\rho}_p^1(J) + \widetilde{\rho}_p^1(K)$

$$(2) \quad \widetilde{\rho}_p^1(J_\eta(K)) = \begin{cases} \widetilde{\rho}_p^1(J) & \text{if } \eta = 0 \text{ in } \widetilde{A}_0^p(J) \\ \widetilde{\rho}_p^1(J) + \rho^0(K) & \text{if } \eta \neq 0 \text{ in } \widetilde{A}_0^p(J) \end{cases}$$

5. STRONGLY p -ANISOTROPIC KNOTS

For a knot K the classical rational Blanchfield form Bl is sesquilinear with respect to the involution $\bar{q}(t) = q(t^{-1})$. For a symmetric polynomial p , this involution extends over \widetilde{R}_p , so the Blanchfield form extends to a sesquilinear form which we call the **strongly localized Blanchfield form**,

$$\widetilde{Bl}_p : \widetilde{A}_0^p(K) \times \widetilde{A}_0^p(K) \rightarrow \frac{\mathbb{Q}(t)}{\widetilde{R}_p}.$$

A submodule P of $\widetilde{A}_0^p(K)$ is called **isotropic** if $P \subseteq P^\perp$ with respect to \widetilde{Bl}_p and it is called **Lagrangian** or **self-annihilating** if $P = P^\perp$. A knot K is called **strongly p -anisotropic** if $\widetilde{A}_0^p(K)$ has no nontrivial isotropic submodules.

We now provide examples of strongly p -anisotropic knots.

Proposition 5.1. *Let Δ be the Alexander polynomial of a knot K . Let p be a symmetric polynomial and $h = (\Delta, p)$ be the greatest common divisor of Δ and p . Suppose that h has no non-symmetric factors and no roots of multiplicity greater than 1. If $(\frac{\Delta}{h}, p) = 1$ then K is strongly p -anisotropic.*

Proof. As a first step, we show that $\widetilde{A}_0^p(K)$ is cyclic. Since the unlocalized Alexander module $A_0(K)$ is torsion over the PID $\mathbb{Q}[t^{\pm 1}]$, it has a decomposition into elementary factors:

$$A_0(K) = \bigoplus_{i=1}^k \frac{\mathbb{Q}[t^{\pm 1}]}{(q_i)}$$

where q_i divides q_{i+1} for each $0 < i < k$ and $\prod_{i=1}^k q_i = \Delta$. Thus, the localized Alexander module has the decomposition

$$\widetilde{A}_0^p(K) = A_0(K) \otimes_{\mathbb{Q}[t^{\pm 1}]} \widetilde{R}_p = \bigoplus_{i=1}^k \frac{\widetilde{R}_p}{(q_i)}$$

If for some $i < k$ there exist some $z \in \mathbb{C}, m, n \in \mathbb{Z}$ such that $q_i(z^n) = p(z^m) = 0$ then $q_{i+1}(z^n)$ is also zero since q_i divides q_{i+1} . Thus, $\Delta = \prod_{i=1}^k q_i$ has a root of multiplicity at least two at z^n . Since h has no roots of multiplicity greater than 1, $\frac{\Delta}{h}(z^n) = 0$ contradicting that $(\frac{\Delta}{h}, p) = 1$. Thus, $q_i \in \widetilde{S}_p$ is a unit in \widetilde{R}_p for every $i < k$ and $\widetilde{A}_0^p(K) = \frac{\widetilde{R}_p}{(q_k)}$. In particular, $\widetilde{A}_0^p(K)$ is cyclic.

Let $\hat{h} = (q_k, h)$. Since $\frac{q_k}{\hat{h}}$ divides $\frac{\Delta}{h}$, which is a polynomial strongly coprime to p , it follows that $\frac{q_k}{\hat{h}} \in \widetilde{S}_p$ is a unit in \widetilde{R}_p so that the ideals generated by \hat{h} and q_k in \widetilde{R}_p are the same and $\widetilde{A}_0^p(K) \cong \widetilde{R}_p/(\hat{h})$.

Let η be a generator of $\widetilde{A}_0^p(K)$. If $\widetilde{Bl}_p(\eta, \frac{x}{y}\eta) = 0$ for some $\frac{x}{y} \in \widetilde{R}_p$, then for any other $\frac{r}{q}\eta \in \widetilde{A}_0^p(K)$, $\widetilde{Bl}_p(\frac{r}{q}\eta, \frac{x}{y}\eta) = \frac{r}{q}\widetilde{Bl}_p(\eta, \frac{x}{y}\eta) = 0$. Thus, $\widetilde{Bl}_p(-, \frac{x}{y}\eta)$ is identically zero. By the non-singularity of the Blanchfield form, this implies that $\frac{x}{y}\eta$ is zero, so that $\frac{x}{y}$ is zero in $\widetilde{R}_p/(\hat{h})$ and $\frac{x}{y} \in (\hat{h})$.

If $\frac{x}{y}\eta$ is an isotropic element of $\widetilde{A}_0^p(K)$, then

$$0 = \widetilde{Bl}_p\left(\frac{x}{y}\eta, \frac{x}{y}\eta\right) = \widetilde{Bl}_p\left(\eta, \frac{x\bar{x}}{y\bar{y}}\eta\right)$$

so that $\frac{x\bar{x}}{y\bar{y}} \in (\hat{h})$. Thus, there is some $\frac{u}{v} \in \widetilde{R}_p$ such that $\frac{x\bar{x}}{y\bar{y}} = \hat{h}\frac{u}{v}$ in \widetilde{R}_p . Cross-multiplying gives the equality in $\mathbb{Q}[t^{\pm 1}]$

$$(5.1) \quad x\bar{x}v = \hat{h}u y\bar{y}.$$

Thus, \hat{h} divides $x\bar{x}v$. The fact that $v \in \widetilde{S}_p$ implies $\widetilde{(v, p)} = 1$ and in particular $(v, p) = 1$. Since \hat{h} divides p it follows that $(v, \hat{h}) = 1$. Therefore (5.1) implies that \hat{h} divides $x\bar{x}$. Since \hat{h} divides h , which has neither any non-symmetric factors nor any repeated factors, it must be that \hat{h} has neither any non-symmetric factors nor any repeated factors. Thus, \hat{h} dividing $x\bar{x}$ implies that \hat{h} divides x so that $\frac{x}{y}$ is in the ideal of \widetilde{R}_p generated by \hat{h} and $\frac{x}{y}\eta = 0$.

Thus, $\widetilde{A}_0^p(K)$ has no nonzero isotropic submodules with respect to \widetilde{Bl}_p . \square

We restrict Proposition 5.1 to the setting from which we draw examples:

Corollary 5.2. *Let Δ be the Alexander polynomial of a knot K .*

- (1) *If $\widetilde{(p, \Delta)} = 1$ then $\widetilde{A}_0^p(K) = 0$ and K is strongly p -anisotropic.*
- (2) *If $p = \Delta$ has no repeated roots and has no non-symmetric factors then K is strongly p -anisotropic.*

6. ISOTROPY AND EXTENSION OF COEFFICIENT SYSTEMS

In this section we discuss the affect that the conditions of strong p -anisotropy and double anisotropy have on higher order Alexander modules of a knot. More precisely, if a knot is strongly p -anisotropic we discover a restriction on the structure of isotropic submodules of certain localizations of higher order Alexander modules. We go on to show that doubly anisotropic elements of the Alexander module produce elements of unlocalized higher order Alexander modules which are almost doubly anisotropic.

We begin by describing what we mean by higher order Alexander modules.

For a knot, K . Let $\psi : \pi_1(M(K)) \rightarrow \Gamma$ be a homomorphism to a PTFA group, Γ , which factors as

$$\pi_1(M(K)) \rightarrow \langle t \rangle \hookrightarrow A \trianglelefteq \Gamma$$

where t is the generator of the Abelianization of $\pi_1(M(K))$ and A is a TFA normal subgroup of Γ . Let

$$S_p(A) = \{q_1(a_1) \dots q_n(a_n) | \widetilde{(q_i, p)} = 1, a_i \in A\} \subseteq \mathbb{Q}[A].$$

Since A is normal, $S_p(A)$ is a Γ -invariant divisor set for $\mathbb{Q}[A]$, [9, Proposition 4.1] shows that $S_p(A) \subseteq \mathbb{Q}[\Gamma]$ satisfies the right (as well as the left) Ore condition and the localization $\mathcal{R} := \mathbb{Q}[\Gamma]S_p(A)^{-1}$ is defined. For the definition of the Ore condition and a treatment of localization for noncommutative rings, see [21, Chapter 2]. Let $\mathcal{K}(\Gamma) = \mathbb{Q}[\Gamma](\mathbb{Q}[\Gamma] - \{0\})^{-1}$ be the skew field of fractions of $\mathbb{Q}[\Gamma]$.

We are interested in the localized higher order Alexander module of K , $H_1(M(K); \mathcal{R})$.

According to [10, Theorem 2.13], there exists a sesquilinear form

$$Bl_\Gamma : H_1(M(K); \mathcal{R}) \times H_1(M(K); \mathcal{R}) \rightarrow \frac{\mathcal{K}(\Gamma)}{\mathcal{R}}.$$

We summarize the construction. Consider the Bockstien exact sequence on cohomology,

$$\begin{aligned} H^1(M(K); \mathcal{K}(\Gamma)) &\rightarrow H^1\left(M(K); \frac{\mathcal{K}(\Gamma)}{\mathcal{R}}\right) \xrightarrow{Bo} \\ H^2(M(K); \mathcal{R}) &\rightarrow H^2(M(K); \mathcal{K}(\Gamma)). \end{aligned}$$

By Poincaré duality and [6, Lemma 3.9],

$$H^2(M(K); \mathcal{K}(\Gamma)) \cong H_1(M(K); \mathcal{K}(\Gamma)) = 0$$

By [10, Remark 2.8.1] there is a universal coefficient theorem for skew field coefficients and

$$H^1(M(K); \mathcal{K}(\Gamma)) \cong \text{Hom}_{\mathcal{K}(\Gamma)}(H_1(M(K); \mathcal{K}(\Gamma)), \mathcal{K}(\Gamma)) = 0.$$

What remains of the Bockstien exact sequence is that the Bockstien homomorphism $Bo : H^1(M(K); \mathcal{K}(\Gamma)/\mathcal{R}) \rightarrow H^2(M(K); \mathcal{R})$ is an isomorphism. The Blanchfield form, Bl_Γ , is defined by the composition

$$\begin{aligned} H_1(M(K); \mathcal{R}) &\xrightarrow{P.D.} H^1(M(K); \mathcal{R}) \xrightarrow{Bo^{-1}} H^1(M(K); \mathcal{K}(\Gamma)/\mathcal{R}) \\ &\xrightarrow{\kappa} \text{Hom}_{\mathcal{R}}(H_1(M(K); \mathcal{R}), \mathcal{K}(\Gamma)/\mathcal{R}) \end{aligned}$$

where $P.D.$ denotes Poincaré duality and κ is the Kronecker map, that is,

$$Bl_\Gamma(a, b) = ((\kappa \circ Bo^{-1} \circ P.D.)(a))(b)$$

By [17, Lemma 3.2 and Proposition 3.6], since $\pi_1(M(K)) \rightarrow \Gamma$ factors nontrivially through Abelianization and has image in the normal TFA subgroup A ,

$$H_1(M(K); \mathcal{R}) \cong H_1(M; \widetilde{R_p}) \otimes_{\widetilde{R_p}} \mathcal{R}$$

and for any $a \otimes \alpha$ and $b \otimes \beta$ in $H_1(M(K); \mathbb{Q}[\Gamma]S_p(A)^{-1})$,

$$Bl_\Gamma(a \otimes \alpha, b \otimes \beta) = \bar{\alpha}\Psi(\widetilde{Bl}_p(a, b))\beta,$$

where $\Psi : \frac{\mathbb{Q}(t)}{\widetilde{R_p}} \rightarrow \frac{\mathcal{K}(\Gamma)}{\mathcal{R}}$ is the map induced by ψ .

Thus, in this section we start with a torsion $\widetilde{R_p}$ module, M with a bilinear form $B : M \times M \rightarrow \frac{\mathbb{Q}(t)}{\widetilde{R_p}}$ and study the bilinear form on $M_\Gamma = M \otimes_{\widetilde{R_p}} \mathcal{R}$,

$B_\Gamma : M_\Gamma \times M_\Gamma \rightarrow \frac{\mathcal{K}(\Gamma)}{\mathcal{R}}$ given by $B_\Gamma(a \otimes \alpha, b \otimes \beta) = \bar{\alpha}\Psi(B(a, b))\beta$.

6.1. The inheritance of anisotropy under extension of coefficients.

The following theorem reveals an aspect of the behavior of isotropic submodules under this extension of coefficients.

Theorem 6.1. *Consider the infinite cyclic group $\langle t \rangle$ and a torsion $\widetilde{R_p}$ module M , with bilinear form $B : M \times M \rightarrow \frac{\mathbb{Q}(t)}{\widetilde{R_p}}$.*

Suppose $\langle t \rangle$ injects into a TFA group A which is a normal subgroup of a PTFA group Γ . If P is an isotropic submodule of $M \otimes \mathcal{R}$ with respect to B_Γ , then $\{m \in M \mid m \otimes 1 \in P\}$ is isotropic with respect to B .

Theorem 6.1 is a consequence of Lemma 6.2 below.

Lemma 6.2. *Suppose $t \mapsto T \in A$ defines a monomorphism from $\langle t \rangle$ to A where A is a TFA group and a normal subgroup of a PTFA group Γ . Then the induced map*

$$\Psi : \frac{\mathbb{Q}(t)}{\widetilde{R_p}} \hookrightarrow \frac{\mathcal{K}(\Gamma)}{\mathcal{R}}.$$

is a monomorphism.

Proof. Suppose that $\frac{f(t)}{g(t)}$ is in the kernel of this map. Then $\frac{f(T)}{g(T)}$ is contained in \mathcal{R} and there exist some $r \in \mathbb{Q}[\Gamma]$ and $q \in S_p(A)$ such that $\frac{f(T)}{g(T)} = \frac{r}{q}$. By the definition of equality in $\mathcal{K}(\Gamma)$ (see [21, Chapter 2 Proposition 1.4]) This implies that there exist nonzero $c, d \in \mathbb{Q}[\Gamma]$ such that

$$(6.1) \quad f(T)c = rd,$$

$$(6.2) \quad g(T)c = qd.$$

Considering (6.1) as an equation in $\mathbb{Q}[\Gamma](\mathbb{Q}[A] - \{0\})^{-1}$ (into which $\mathbb{Q}[\Gamma]$ injects), it reduces to $c = (f(T))^{-1}rd$. This substitution reduces (6.2) to $g(T)(f(T))^{-1}rd = qd$. Cancellation gives that $g(T)(f(T))^{-1}r = q$. Now, $g(T)$ and $(f(T))^{-1}$ sit in the image of the field $\mathbb{Q}(t)$ and so commute with each other. Thus, $(f(T))^{-1}g(T)r = q$. Multiplying by $f(T)$ on the left gives us that

$$(6.3) \quad g(T)r = f(T)q.$$

Let X be a transversal for Γ/A (that is, a subset of Γ containing the identity, 1, such that every equivalence class in Γ/A has a unique representative in X). The group ring $\mathbb{Q}[\Gamma]$ is free as a $\mathbb{Q}[A]$ module and has basis X , so that r can be uniquely realized as $r = \sum_{x \in X} r_x x$ where each r_x is in $\mathbb{Q}[A]$. Since the right hand side of (6.3) is in $\mathbb{Q}[A]$, that is, the span of $\{1\}$, it reduces to

$$(6.4) \quad g(T)r_1 = f(T)q \text{ and}$$

$$(6.5) \quad r_x = 0 \text{ for } x \in X - \{1\}.$$

Now suppose that $\frac{f(t)}{g(t)} \in \mathbb{Q}(t)$ is in reduced terms and that g is not in \widetilde{S}_p . Since q is assumed to be in $S_p(A)$, there exist polynomials $q_1, \dots, q_n \in \widetilde{S}_p$ and $a_1, \dots, a_n \in A$ so that $q = \prod_{i=1}^n q_i(a_i)$. Equation (6.4) as an equality in $\mathbb{Q}[A]$ involves only finitely many elements of A , namely, T, a_1, \dots, a_n and b_1, \dots, b_k where b_1, \dots, b_k are whatever terms appear in r_1 . The span of $\{T, a_1, \dots, a_n, b_1, \dots, b_k\}$ is a finitely generated subgroup of the TFA group, A , so it is free Abelian. Pick a basis $\{s, c_1, \dots, c_m\}$ such that $s^l = T$ for some l . The equality (6.4) can then be realized as an equality in the multivariable Laurent polynomial ring $\mathbb{Q}[s^{\pm 1}, c_1^{\pm 1}, \dots, c_m^{\pm 1}]$:

$$(6.6) \quad g(s^k)r_1(s, c_1, \dots, c_m) = f(s^k) \prod_{i=1}^n q_i(s^{k_i} c_1^{k_{i,1}} \dots c_m^{k_{i,m}}).$$

Since g is not an element of \widetilde{S}_p , (i.e. it is not strongly coprime to p) there exists some $z \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{Z} - \{0\}$ such that $g(z^\beta) = p(z^\alpha) = 0$. Evaluating (6.6) at $s = z^{\beta/k}$ and $c_1 = \dots = c_n = 1$ gives an equality in \mathbb{C}

$$(6.7) \quad 0 = f(z^\beta) \prod_{i=1}^n q_i(z^{\beta k_i/k}).$$

Since z^β is a root of g and f is assumed to be relatively prime to g , $f(z^\beta) \neq 0$. Therefore $q_i(z^{\beta k_i/k}) = 0$ for some i , contradicting that $q_i \in \widetilde{S}_p$ is strongly coprime to p . Thus, it must be that $g \in \widetilde{S}_p$ and $\frac{f}{g} = 0$ in $\mathbb{Q}(t)/\widetilde{R}_p$. \square

We now prove Theorem 6.1.

Proof of Theorem 6.1. Consider any $m, n \in M$ such that $m \otimes 1, n \otimes 1 \in P$. Since P is isotropic, $0 = B_\Gamma(m \otimes 1, n \otimes 1) = \Psi(B(m, n))$. Since Ψ is injective by Lemma 6.2 this implies that $B(m, n) = 0$ completing the proof. \square

6.2. The inheritance of double anisotropy under extension of coefficients. The following Proposition reveals that the double anisotropy is partially inherited under the extension of coefficients. A similar claim could be proven after localization. For our purposes the unlocalized claim is sufficient.

Proposition 6.3. *Let A be a TFA group which is a normal subgroup of a PTFA group Γ . Let M be a torsion $\mathbb{Q}[A]$ module with $\mathbb{Q}[A]$ -sesquilinear form $B_A : M \times M \rightarrow \frac{\mathcal{K}(A)}{\mathbb{Q}[A]}$. This sesquilinear form extends to a $\mathbb{Q}[\Gamma]$ -sesquilinear form $B_\Gamma : M \otimes \mathbb{Q}[\Gamma] \times M \otimes \mathbb{Q}[\Gamma] \rightarrow \frac{\mathcal{K}(\Gamma)}{\mathbb{Q}[\Gamma]}$.*

Let Q be an isotropic submodule of M with respect to B_A and P be an isotropic submodule of $M \otimes \mathbb{Q}[\Gamma]$ with respect to B_Γ . If $\eta \otimes 1 \in M \otimes \mathbb{Q}[\Gamma]$ sits in $P + Q \otimes \mathbb{Q}[\Gamma]$, then $\eta = p + q$ for some $q \in Q$ and $B_A(p, p) = 0$.

Proof. Let X be a transversal for Γ/A . Suppose that

$$(6.8) \quad \eta \otimes 1 = \sum_{x \in X} p_x \otimes x + \sum_{x \in X} q_x \otimes x$$

with $q_x \in Q$, $p_x \in M$ for all x and $p := \sum_{x \in X} p_x \otimes x \in P$. Thinking of $\mathbb{Q}[\Gamma]$ as the free $\mathbb{Q}[A]$ module generated by X , (6.8) implies that $p_x + q_x = 0$ for $x \neq 1$ and $\eta = p_1 + q_1$. Since P is isotropic,

$$(6.9) \quad 0 = B_\Gamma(p, p) = B_\Gamma \left(\sum_{x \in X} p_x \otimes x, \sum_{y \in X} p_y \otimes y \right).$$

Appealing to the Γ -sesquilinearity of B_Γ , this implies

$$(6.10) \quad \begin{aligned} 0 &= \sum_{x \in X} \sum_{y \in X} x^{-1} B_\Gamma(p_x \otimes 1, p_y \otimes 1) y \\ &= \sum_{x \in X} \sum_{y \in X} (x^{-1} B_A(p_x, p_y) x) x^{-1} y. \end{aligned}$$

Since A is normal in Γ , $x^{-1} B_A(p_x, p_y) x$ is in $\frac{\mathcal{K}(A)}{\mathbb{Q}[A]}$ for each $x, y \in X$.

Since X is a choice of coset representatives for Γ/A , each $x^{-1}y$ is equivalent modulo A to some z in X , that is, there is some $a_{x,y} \in A$ such that $x^{-1}y = a_{x,y}z$. We use this to rearrange (6.10),

$$(6.11) \quad 0 = \sum_{z \in X} \sum_{x^{-1}y \equiv z} (x^{-1} B_A(p_x, p_y) x) a_{x,y} z$$

The map $\left(\frac{\mathcal{K}(A)}{\mathbb{Q}[A]} \right)^{|X|} \rightarrow \frac{\mathcal{K}(\Gamma)}{\mathbb{Q}[\Gamma]}$ defined by sending $\langle r_x \rangle_{x \in X}$ to $\sum_{x \in X} r_x \otimes x$ is injective. Indeed, if $\langle \frac{a_x}{b_x} \rangle_{x \in X}$ is in the kernel of this map then there exists some $c = \sum_{x \in X} c_x x$ with $c_x \in \mathbb{Q}[A]$ such that

$$\sum_{x \in X} a_x b_x^{-1} x = \sum_{x \in X} c_x x$$

Since A is Abelian, $a_x b_x^{-1} = b_x^{-1} a_x$. Left multiplying by $b = \prod_{x \in X} b_x$ we see that

$$\sum_{x \in X} (b_x^{-1} b) a_x x = \sum_{x \in X} b c_x x$$

This is an equation in $\mathbb{Q}[\Gamma]$. The set X is a basis for $\mathbb{Q}[\Gamma]$ as a free $\mathbb{Q}[A]$ module. Thus, for all $x \in X$, $(b_x^{-1} b) a_x = b c_x$ so $\frac{a_x}{b_x} = c_x \in \mathbb{Q}[A]$ holds in

$\mathbb{Q}(A)$ and $\langle \frac{a_x}{b_x} \rangle_{x \in X}$ is zero in $\left(\frac{\mathcal{K}(A)}{\mathbb{Q}[A]} \right)^{|X|}$.

This injectivity together with (6.11) implies that for each $z \in X$,

$$(6.12) \quad 0 = \sum_{x^{-1}y \equiv z} (x^{-1} B_A(p_x, p_y) x) a_{x,y}.$$

Taking $z = 1 \in X$ we see that

$$(6.13) \quad 0 = \sum_{x \in X} (x^{-1} B_A(p_x, p_x) x) a_{x,x}.$$

As we observed previously, for $x \neq 1$, $p_x = -q_x$ is in Q , so $B_A(p_x, p_x) = 0$. Thus, all but one of the terms in (6.13) vanishes. Dropping them, we see that $0 = B_A(p_1, p_1)$. Since, $\eta = p_1 + q_1$, this completes the proof. \square

Proposition 6.4. *Let M be a torsion $\mathbb{Q}[t^{\pm 1}]$ -module with bilinear form $B : M \times M \rightarrow \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]}$. Let $Q \subseteq M$ be isotropic.*

Suppose $\langle t \rangle$ injects into a TFA group A which is a normal subgroup of a PTFA group Γ and $P \subseteq M \otimes \mathbb{Q}[\Gamma]$ is isotropic. If $\eta \otimes 1$ is in $P + Q \otimes \mathbb{Q}[\Gamma]$, then η is not doubly anisotropic with respect to B .

Proof. Consider $\eta^A = \eta \otimes 1 \in M \otimes \mathbb{Q}[A]$ and the isotropic module $Q^A = Q \otimes \mathbb{Q}[A]$. Notice that $\eta^A \otimes 1 \in (M \otimes \mathbb{Q}[A]) \otimes \mathbb{Q}[\Gamma]$ is in $Q^A \otimes \mathbb{Q}[\Gamma] + P$ by assumption. Applying Proposition 6.3 gives that there is some $q^A \in Q^A$ and $p^A \in M \otimes \mathbb{Q}[A]$ with $B_A(p^A, p^A) = 0$ and $\eta^A = p^A + q^A$.

But then $\eta \otimes 1 \in M \otimes \mathbb{Q}[A]$ sits in the sum of $Q \otimes \mathbb{Q}[A]$ with the isotropic submodule $\langle p^A \rangle$. Applying Proposition 6.3 again gives that $\eta = p + q$ where $B(p, p) = 0$ and $q \in Q$. In this case, η sits in the sum of the isotropic submodules Q and $\langle p \rangle$ and so is not doubly anisotropic. \square

7. THE PROOF OF THEOREM 1.1

In this section we set out to prove the main result of this paper, Theorem 1.1:

Theorem 1.1. *Let $\{K_i\}$ be a possibly infinite set of knots:*

- (1) *whose Alexander polynomials are strongly coprime,*
- (2) *whose Tristram-Levine signatures have vanishing integrals,*
- (3) *whose prime factors have square-free Alexander polynomials and*
- (4) *whose ρ^1 -invariants do not vanish, that is $\rho^1(K_i) \neq 0$.*

For $i = 1, 2, \dots$ and $j = 1, 2, \dots, n$ let $R_{i,j}$ be a slice knot and $\eta_{i,j}$ be an unknotted curve in the complement of $R_{i,j}$ such that the pair $(R_{i,j}, \eta_{i,j})$ is doubly anisotropic.

Let $K_i^0 = K_i$ and $K_i^j = R_{i,j}(\eta_{i,j}, K_i^{j-1})$.

Then $\{K_i^n\}_{i=1}^\infty$ is linearly independent modulo $(n + 1.5)$ -solvable knots.

In order to prove the theorem we explore the interaction between an iterated infection procedure, strongly localized ρ -invariants and the following technical condition on a bounded 4-manifold.

Definition 7.1. Let K_1, \dots, K_m be knots in S^3 . Consider a 4-manifold W with $\partial W = \sqcup M(K_i)$ and an epimorphism $\phi : \pi_1(W) \twoheadrightarrow \Gamma$. The pair (W, Γ) is said to satisfy **condition C** with respect to integers n, h (which we abbreviate by saying that (W, Γ) is **C(n, h)**) if the following conditions hold:

- (C1) $\Gamma_{\mathfrak{r}}^{(n+1)} = 0$ (This condition implies that Γ is PTFA).
- (C2) There is a normal Abelian subgroup $A \triangleleft \Gamma$ such that for each i , there is a monomorphism $\alpha_i : \frac{\pi_1(M(K_i))}{\pi_1(M(K_i))_{\mathfrak{r}}^{(1)}} \cong \mathbb{Z} \rightarrow \Gamma$ making the following diagram commute,

$$\begin{array}{ccc} \pi_1(M(K_i)) & \longrightarrow & \frac{\pi_1(M(K_i))}{\pi_1(M(K_i))_{\mathfrak{r}}^{(1)}} \\ \downarrow & & \downarrow \alpha_i \\ \pi_1(W) & \xrightarrow{\phi} & \Gamma \end{array},$$

and $\text{Im}(\alpha_i)$ sits in A . The subgroup A does not depend on i .

- (C3) For any coefficient system $\psi : \pi_1(W) \rightarrow \Lambda$ with $\Lambda_{\mathfrak{r}}^{(h+1)} = 1$ and an epimorphism, β , making the following diagram commute

$$\begin{array}{ccc} \pi_1(W) & \xrightarrow{\phi} & \Gamma \\ & \searrow \psi & \uparrow \beta \\ & & \Lambda \end{array}$$

and for any Ore localization $\mathbb{Q}[\Lambda]S^{-1}$ of $\mathbb{Q}[\Lambda]$,

$$\text{Ker}(H_1(\partial W; \mathbb{Q}[\Lambda]S^{-1}) \rightarrow H_1(W; \mathbb{Q}[\Lambda]S^{-1}))$$

is isotropic with respect to the Blanchfield form

$$Bl_{\Lambda} : H_1(\partial W; \mathbb{Q}[\Lambda]S^{-1}) \times H_1(\partial W; \mathbb{Q}[\Lambda]S^{-1}) \rightarrow \frac{\mathcal{K}(\Lambda)}{\mathbb{Q}[\Lambda]S^{-1}}.$$

- (C4) For any PTFA coefficient system $\psi : \pi_1(W) \rightarrow \Theta$ on W , with $\Theta_{\mathfrak{r}}^{(h+2)} = 0$, $\sigma^{(2)}(W; \psi) - \sigma(W) = 0$

If such a pair (W, Γ) exists then we say that $\sqcup M(K_i)$ bounds a $C(n, h)$.

Remark 7.2. Notice that if $n \leq h$, then taking $\Lambda = \Gamma$, β to be the identity map and A to be the Abelian group of condition (C2), then condition (C3) implies that for a polynomial, p ,

$$\text{Ker} (H_1(\partial W; \mathbb{Q}[\Gamma]S_p(A)^{-1}) \rightarrow H_1(W; \mathbb{Q}[\Gamma]S_p(A)^{-1}))$$

is isotropic with respect to the Blanchfield form on $H_1(\partial W; \mathbb{Q}[\Gamma]S_p(A)^{-1})$.

Notice that $(h.5)$ -solutions satisfy this condition.

Lemma 7.3. *If W is an $h.5$ solution for K and $\phi : \pi_1(W) \rightarrow \mathbb{Z}$ is the Abelianization map then (W, \mathbb{Z}) is $C(0, h - 1)$.*

Proof. Condition (C1) holds since \mathbb{Z} is torsion free Abelian. Condition (C2) holds since by definition of $h.5$ solvability [10, Definition 1.2] the inclusion induced map on first homology is an isomorphism. Condition (C3) follows from [7, Theorem 6.3] since W being an $(h.5)$ -solution implies that it is an (h) -solution and so is an (h) -null-bordism. Condition (C4) follows from [10, Theorem 4.2]. \square

The action of connected sum and infection on Condition C are provided by the following lemmas. Their proofs are delayed until subsection 7.1.

Lemma 7.4. *Let $K_i = K_{i,1} \# \dots \# K_{i,m_i}$ for $i = 1, \dots, p$. If $\bigsqcup_{i=1}^p M(K_i)$ bounds a $C(n, h)$, (W, Γ) , then $\bigsqcup_{i=1}^p \bigsqcup_{j=1}^{m_i} M(K_{i,j})$ bounds a $C(n, h)$.*

Lemma 7.5. *For $1 \leq i \leq m$, let K_i be a knot, R_i be a slice knot and η_i be an unknotted curve representing a doubly anisotropic element of $A_0(R_i)$. If $\bigsqcup_{i=1}^m M(R_i(\eta_i, K_i))$ bounds a $C(n, h)$ with $n \leq h$, then $\bigsqcup_{i=1}^m M(K_i)$ bounds a $C(n + 1, h)$.*

We now combine the three lemmas above to discover a relationship between the result of iterated infection being solvable and zero surgery on the deepest infecting knots cobounding a 4-manifold satisfying condition C .

Lemma 7.6. *For integers $1 \leq i \leq m$ and $1 \leq j \leq n$ let $R^{i,j}$ be a slice knot with doubly anisotropic curve $\eta_{i,j}$. For $1 \leq i \leq m$ let K_i be a knot. Let $K^{i,j}$ be recursively defined by $K^{i,0} = K_i$ and $K^{i,j} = R_{\eta_{i,j}}^{i,j}(K^{i,j-1})$. If $\bigsqcup_{i=1}^m K^{i,n}$ is $(h.5)$ -solvable for $h \geq n$, then $\bigsqcup_{i=1}^m M(K_i)$ bounds a 4-manifold with coefficient system Γ such that (W, Γ) is $C(n, h - 1)$.*

Proof. If $\bigsqcup_{i=1}^m K^{i,n}$ is $(h.5)$ -solvable then Lemma 7.3 implies that $M\left(\bigsqcup_{i=1}^m K^{i,n}\right)$ bounds a $C(0, h - 1)$. Applying Lemma 7.4, this means that $\bigsqcup_{i=1}^m M(K^{i,n})$ bounds a $C(0, h - 1)$.

Now, applying Lemma 7.5 (if $h - 1 \geq 0$) implies that $\bigsqcup_{i=1}^m M(K^{i,n-1})$ bounds a $C(1, h-1)$. Applying it again (if $h-1 \geq 1$) implies that $\bigsqcup_{i=1}^m M(K^{i,n-2})$ bounds a $C(2, h-1)$. Applying it a total of n times (provided that $h-1 \geq n-1$) gives that $\bigsqcup_{i=1}^m M(K^{i,n-n}) = \bigsqcup_{i=1}^m M(K_i)$ bounds a $C(n, h-1)$, as claimed. \square

Next, we prove that the $\tilde{\rho}_p^1$ -invariant is an obstruction to a disjoint union of zero surgeries on knots bounding a 4-manifold satisfying condition C .

Lemma 7.7. *Let $p \in \mathbb{Q}[t^{\pm 1}]$ be a polynomial. If $\{K_i\}_{i=1}^m$ is a set of knots such that for each i , K_i decomposes as a connected sum of p -anisotropic knots and $\bigsqcup_{i=1}^m M(K_i)$ bounds a 4-manifold W with coefficient system Γ such*

that (W, Γ) is $C(n, h)$ with $n \leq h$, then $\sum_{i=1}^m \tilde{\rho}_p^1(K_i) = 0$.

Proof. We first prove the lemma in the more restrictive setting that each K_i is strongly p -anisotropic. By (C2), the coefficient system $\phi : \pi_1(W) \rightarrow \Gamma$ restricted to the $M(K_i)$ -boundary component factors nontrivially through the Abelianization of $\pi_1(M(K_i))$ and by (C4) the associated signature defect is zero. Thus,

$$\sum_i \rho^0(K_i) = 0.$$

While noteworthy, this is not the desired conclusion. We find another coefficient system on W which, when restricted to each $M(K_i)$ boundary component, factors injectively through the quotient of $\pi_1(M(K_i))$ by $\pi_1(M(K_i))_p^{(2)}$.

By Remark 7.2

$$P = \text{Ker}(H_1(M(K_i); \mathbb{Q}[\Gamma]S_p(A)^{-1}) \rightarrow H_1(W; \mathbb{Q}[\Gamma]S_p(A)^{-1}))$$

is isotropic.

Since $\pi_1(M(K)) \rightarrow \Gamma$ factors nontrivially through the Abelianization, [17, Lemma 3.2 and Proposition 3.6] reveals that

$$H_1(M(K_i); \mathbb{Q}[\Gamma]S_p(A)^{-1}) \cong H_1(M(K_i); \widetilde{R_p}) \otimes_{\widetilde{R_p}} \mathbb{Q}[\Gamma]S_p(A)^{-1},$$

and for any $a \otimes \alpha$ and $b \otimes \beta$ in $H_1(M(K_i); \widetilde{R_p}) \otimes_{\widetilde{R_p}} \mathbb{Q}[\Gamma]S_p(A)^{-1}$,

$$Bl_\Gamma(a \otimes \alpha, b \otimes \beta) = \bar{\alpha} \Psi(\widetilde{Bl_p}(a, b)) \beta,$$

where $\Psi : \frac{\mathbb{Q}(t)}{\widetilde{R_p}} \rightarrow \frac{\mathcal{K}(\Gamma)}{\widetilde{R_p}}$ is induced by the map α_i of condition (C2). Thus, we can think of P as an isotropic submodule of $\widetilde{A_0^p}(K) \otimes \mathbb{Q}[\Gamma]S_p(A)^{-1}$ with respect to the bilinear form induced by $\widetilde{Bl_p}$

By applying Theorem 6.1, we see that the kernel of the map

$$(7.1) \quad \widetilde{A}_0^p(K_i) \rightarrow \widetilde{A}_0^p(K_i) \otimes_{\widetilde{R}_p} \mathbb{Q}[\Gamma]S_p(A)^{-1} = H_1(W; \mathbb{Q}[\Gamma]S_p(A)^{-1}),$$

which is equal to $\{m \in \widetilde{A}_0^p(K_i) | m \otimes 1 \in P\}$ is isotropic. We assume that $\widetilde{A}_0^p(K_i)$ has no nontrivial isotropy so (7.1) is injective.

Now we build a new coefficient system on W . Let $G := \text{Ker}(\phi : \pi_1(W) \rightarrow \Gamma)$, so that G is isomorphic to the fundamental group of \widetilde{W}_Γ , the Γ -cover of W . Let $G_p^{(1)} \subseteq G$ be given by the kernel of the following composition:

$$F : G \xrightarrow{\cong} \pi_1(\widetilde{W}_\Gamma) \rightarrow \frac{H_1(\widetilde{W}_\Gamma; \mathbb{Z})}{\mathbb{Z}\text{-torsion}} \hookrightarrow H_1(W; \mathbb{Q}[\Gamma]) \rightarrow H_1(W; \mathbb{Q}[\Gamma]S_p(A)^{-1})$$

Observe that $G_p^{(1)}$ is normal in $\pi_1(W)$. In order to see this, let $g \in G_p^{(1)}$ and $\gamma \in \pi_1(W)$. Then $F(\gamma^{-1}g\gamma)$ is given by letting γ_* , the deck translation on \widetilde{W}_Γ corresponding to γ , act on $F(g) \in H_1(W; \mathbb{Q}[\Gamma]S_p(A)^{-1})$. Since $F(g) = 0$, it follows that $F(\gamma^{-1}g\gamma) = \gamma_*(F(g)) = 0$, so $\gamma^{-1}g\gamma$ is in $G_p^{(1)}$.

Since (C1) gives us that $\pi_1(M(K_i)) \rightarrow \Gamma$ factors through Abelianization, it follows that ϕ is trivial on $\pi_1(M(K_i))^{(1)}$ and the map induced by inclusion sends $\pi_1(M(K_i))^{(1)}$ to G . Consider the following commutative diagram:

$$(7.2) \quad \begin{array}{ccccc} \pi_1(M(K_i))^{(1)} & \xrightarrow{a} & \frac{\pi_1(M(K_i))^{(1)}}{\pi_1(M(K_i))_{\widetilde{p}}^{(2)}} & \xrightarrow{b} & \widetilde{A}_0^p(K) \\ \downarrow i_* & & \downarrow \beta & & \downarrow c \\ G & \xrightarrow{e} & \frac{G}{G_p^{(1)}} & \xrightarrow{d} & H_1(W; \mathbb{Q}[\Gamma]S_p(A)^{-1}) \end{array}$$

The dotted map, β , is induced by i_* . In order to see that it is well defined, one must check that i_* maps $\pi_1(M(K_i))_{\widetilde{p}}^{(2)}$ to $G_p^{(1)}$. In order to see this, take $x \in \pi_1(M(K_i))_{\widetilde{p}}^{(2)}$. It follows that $c(b(a(x)))$ is zero in $H_1(W; \mathbb{Q}[\Gamma]S_p(A)^{-1})$. By the commutativity of the diagram, $d(e(i_*(x)))$ is zero in $H_1(W; \mathbb{Q}[\Gamma]S_p(A)^{-1})$ and so $i_*(x) \in \text{Ker}(e) = G_p^{(1)}$. Thus, the map β is well defined.

The maps b and d in (7.2) are injections by the definition of $\pi_1(M(K))_{\widetilde{p}}^{(2)}$ and $G_p^{(1)}$. The map c is the monomorphism in (7.1). Thus, β is injective.

If x is in the kernel of the composition

$$(7.3) \quad \pi_1(M(K_i)) \xrightarrow{i_*} \pi_1(W) \rightarrow \frac{\pi_1(W)}{G_p^{(1)}},$$

then $i_*(x)$ is in $G_p^{(1)} \subseteq G := \text{Ker}(\pi_1(W) \rightarrow \Gamma)$. Since the map from $\pi_1(M(K_i))$ to Γ factors nontrivially through Abelianization, it must be that x is in $\pi_1(M(K))^{(1)}$. This means that $x \in \text{Ker} \left(\pi_1(M(K_i))^{(1)} \xrightarrow{i_*} G \rightarrow \frac{G}{G_p^{(1)}} \right)$. By the commutativity of (7.2) and the injectivity of β , the kernel of this map is $\pi_1(M(K_i))_{\tilde{p}}^{(2)}$.

If we set $\Theta := \frac{\pi_1(W)}{G_p^{(1)}}$ then the following commutative diagram holds for each $M(K_i)$ -boundary component:

$$\begin{array}{ccc} \pi_1(M(K_i)) & \longrightarrow & \frac{\pi_1(M(K_i))}{\pi_1(M(K_i))_{\tilde{p}}^{(2)}} \\ \downarrow & & \downarrow \\ \pi_1(W) & \longrightarrow & \Theta. \end{array}$$

This implies that $\sum_{i=1}^m \tilde{\rho}_p^1(K) = \sigma^2(W, \Theta) - \sigma(W)$. It remains only to check that this signature defect is zero.

In order to apply condition (C4) to get this conclusion, consider the following short exact sequence:

$$0 \rightarrow \frac{G}{G_p^{(1)}} \rightarrow \frac{\pi_1(W)}{G_p^{(1)}} \rightarrow \frac{\pi_1(W)}{G} \rightarrow 0$$

The leftmost term, $\frac{G}{G_p^{(1)}}$, is TFA. The rightmost term, $\frac{\pi_1(W)}{G}$, injects into Γ , so that $\left(\frac{\pi_1(W)}{G} \right)_{\mathfrak{r}}^{(n+1)} = 0$ since $\Gamma_{\mathfrak{r}}^{(n+1)} = 1$. This implies that

$$\Theta_{\mathfrak{r}}^{(n+2)} = \left(\frac{\pi_1(W)}{G_p^{(1)}} \right)_{\mathfrak{r}}^{(n+2)} = 0$$

and condition (C4) applies to give that $\sigma^2(W, \Theta) - \sigma(W) = 0$. This completes the proof in the case that each K_i is p -anisotropic.

In order to see it under the weaker assumption that each K_i has only strongly p -anisotropic factors, suppose that $K_i = \#_{b=1}^{B_i} J_{i,b}$ with each $J_{i,b}$ strongly p -anisotropic. An application of Lemma 7.4 gives that

$$\bigsqcup_{i=1}^m \bigsqcup_{b=1}^{B_i} M(J_{i,b}).$$

bounds a $C(n, h)$. Now we can apply the theorem in the case already proven to see that

$$\sum_{i=1}^m \left(\sum_{b=1}^{B_i} \tilde{\rho}_p^1(J_{i,b}) \right) = 0$$

By Proposition 4.3, $\sum_{b=1}^{B_i} \tilde{\rho}_p^1(J_{i,b}) = \tilde{\rho}_p^1(\#_{b=1}^{B_i} J_{i,b}) = \tilde{\rho}_p^1(K_i)$. Making this substitution completes the proof. \square

We are now ready to prove Theorem 1.1. We prove a stronger theorem from which we get it as a corollary.

Theorem 7.8. *Let p be a polynomial. Let $\{K_i\}$ be a possibly infinite set of knots each of which decomposes into a connected sum of strongly p -anisotropic knots. For $i = 1, 2, \dots$ and $j = 1, 2, \dots, n$ let $R_{i,j}$ be a slice knot and $\eta_{i,j}$ be an unknotted curve in the complement of $R_{i,j}$ representing a doubly anisotropic element of $A_0(R_{i,j})$.*

Let $K_i^0 = K_i$ and $K_i^j = R_{i,j}(\eta_{i,j}, K_i^{j-1})$.

If $\#_{i=1}^m a_i K_i^n$ is $(n + 1.5)$ -solvable then $\sum_{i=1}^m a_i \tilde{\rho}_p^1(K_i) = 0$.

Proof. Suppose that $\#_{i=1}^m a_i K_i^n$ were $(n+1.5)$ -solvable. By Lemma 7.6 $\bigsqcup_{i=1}^m \left(\bigsqcup_{k=1}^{a_i} M(K_i) \right)$ bounds a $C(n, n)$ so by Lemma 7.7 $\sum_i a_i \tilde{\rho}_p^1(K_i) = 0$. \square

Proof of Theorem 1.1. Suppose that some linear combination $\#_{j=1}^m a_j K_j$ is $(n + 1.5)$ -solvable. Let p be the Alexander polynomial of K_i . For $j \neq i$ K_j has Alexander polynomial strongly coprime to p . By Corollary 5.2, K_j is strongly p -anisotropic for all j . Theorem 7.8 gives that

$$(7.4) \quad \sum_j a_j \tilde{\rho}_p^1(K_j) = 0$$

Proposition 4.2 applies to give that $\tilde{\rho}_p^1(K_i) = \rho^1(K_i) \neq 0$ and that for $j \neq i$, $\tilde{\rho}_p^1(K_j) = \rho^0(K_j) = 0$. plugging these into (7.4) yields $a_i \rho^1(K_i) = 0$ so $a_i = 0$.

Since the choice of i was arbitrary, $a_i = 0$ for all i and there are no nontrivial linear relationships amongst these knots modulo $n+1.5$ solvability. \square

7.1. Proofs of Lemmas 7.4 and 7.5. Before we prove these two important lemmas we discuss the cobordisms used to prove them

Definition 7.9. For knots K_1, \dots, K_n , let $V_\# = V_\#(K_1, \dots, K_n)$ be the cobordism between $\sqcup M(K_i)$ and $M(\#K_i)$ constructed by starting with $\bigsqcup_{i=1}^n M(K_i) \times [0, 1]$ and connecting it by gluing together neighborhoods of curves in $M(K_{i-1}) \times \{1\}$ and $M(K_i) \times \{1\}$ representing the meridians of K_{i-1} and K_i .

Definition 7.10. Consider knots K and J and an unknotted curve, η , in $S^3 - K$ which has zero linking with K . Let $V_{\text{inf}} = V_{\text{inf}}(K, \eta, J)$ be the cobordism between $M(K) \sqcup M(J)$ and $M(K_\eta(J))$ given by starting with $M(K) \times I \sqcup M(J) \times I$ and gluing a neighborhood of η in $M(K) \times \{1\}$ to a neighborhood of the meridian of J in $M(J) \times \{1\}$.

By virtue of the fact that the inclusion induced maps $H_2(M(K)) \oplus H_2(M(J)) \rightarrow H_2(V_\#)$ and $H_2(M(R)) \oplus H_2(M(K)) \rightarrow H_2(V_{\text{inf}})$ are epimorphisms, each of $V_\#$ and V_{inf} are rational (k) -null-bordisms for every nonnegative integer, k (see [7, Definition 5.1]).

The following is a key result about rational (k) -null-bordisms. (For an \mathcal{R} module M , $T(M)$ denotes the \mathcal{R} -torsion part of M .)

Theorem ([7, Theorem 6.3]). *Suppose W is a rational (k) -null-bordism and $\phi : \pi_1(W) \rightarrow \Gamma$ is a nontrivial coefficient system where Γ is a PTFA group with $\Gamma^{(k)} = 1$. Let \mathcal{R} be an Ore localization of $\mathbb{Z}[\Gamma]$ so $\mathbb{Z}[\Gamma] \subseteq \mathcal{R} \subseteq \mathcal{K}(\Gamma)$. Suppose that for each component M_i of ∂W on which ϕ is nontrivial, $\text{Rank}_{\mathbb{Z}[\Gamma]}(H_1(M_i; \mathbb{Z}[\Gamma])) = \beta_1(M_i) - 1$. Then if P is the kernel of the inclusion induced map $T(H_1(\partial W; \mathcal{R})) \rightarrow T(H_1(W; \mathcal{R}))$ then P is isotropic with respect to the Blanchfield form on $T(H_1(\partial W; \mathcal{R}))$.*

Now, for any PTFA group, Γ , $\Gamma^{(k)} = 1$ for some k . Since the cobordisms in which we are interested are (k) -null bordisms for every k , this condition imposes only the restriction that Γ be PTFA in our setting. Since the components of V_{inf} and $V_\#$ are all given by zero surgery along knots. The condition that $\text{Rank}_{\mathbb{Z}[\Gamma]}(H_1(M_i; \mathbb{Z}[\Gamma])) = \beta_1(M_i) - 1 = 0$ holds for all coefficient systems which are nontrivial on M_i by [6, Proposition 3.10].

In the case of $V_\#$, the meridian of any one of the components normally generates $\pi_1(V_\#)$ so that if ϕ is nontrivial on $V_\#$ then it is nontrivial on every boundary component. In the case of V_{inf} , the meridians of R and $R_\eta(K)$ each normally generate $\pi_1(V_{\text{inf}})$ so that ϕ is nontrivial on the $M(R)$ and $M(R_\eta(K))$ boundary components

Proposition 7.11. *Consider knots K and J . Let η be an unknotted curve in the complement of K . Let V be either $V_\#(K, J)$ or $V_{\text{inf}}(K, \eta, J)$. Given a nontrivial PTFA coefficient system $\phi : \pi_1(V_\#) \rightarrow \Gamma$ and $S \subseteq \mathbb{Q}[\Gamma]$ a right divisor set which is closed under the involution on $\mathbb{Q}[\Gamma]$. Let $\mathcal{R} = \mathbb{Q}[\Gamma]S^{-1}$. Then*

- (1) $\sigma^{(2)}(V, \Gamma) = \sigma(V) = 0$,
- (2) $\text{Ker}(H_1(\partial V_\#; \mathcal{R}) \rightarrow H_1(V_\# \mathcal{R}))$ is isotropic with respect to Bl_Γ .

(3) If $V = V_{\inf}$ and ϕ is nontrivial on $M(J)$, then

$$\text{Ker} (H_1 (\partial V_{\inf}; \mathcal{R}) \rightarrow H_1 (V_{\#}; \mathcal{R}))$$

is isotropic with respect to Bl_{Γ} .

(4) If ϕ is trivial on $M(J)$, then

$$\text{Ker} (H_1 (M(K(\eta, J)); \mathcal{R}) \oplus H_1 (M(K); \mathcal{R}) \rightarrow H_1 (V_{\inf}; \mathcal{R}))$$

is isotropic with respect to Bl_{Γ} .

Proof. By [7, Theorem 5.9], $\sigma^{(2)}(W, \Gamma) = \sigma(W) = 0$.

The remaining conclusions are immediate consequences of [7, Theorem 6.3]. Conclusions (2) and (3) follow since $H_2(\partial V; \mathcal{R})$ is torsion in these cases. Conclusion (4) holds since if ϕ is trivial on $M(J)$, then the torsion part of $H_2(\partial V_{\inf}; \mathcal{R})$ is $H_1 (M(K(\eta, J)); \mathcal{R}) \oplus H_1 (M(K); \mathcal{R})$. \square

Finally, we prove the technical lemmas needed in the proof of Lemma 7.6. For convenience we restate them as we prove them.

Lemma 7.4. *Let $K_i = K_{i,1} \# \dots \# K_{i,m_i}$ for $i = 1, \dots, p$. If $\bigsqcup_{i=1}^p M(K_i)$ bounds a $C(n, h)$, (W, Γ) , then $\bigsqcup_{i=1}^p \bigsqcup_{j=1}^{m_i} M(K_{i,j})$ bounds a $C(n, h)$.*

Proof. Construct a new 4-manifold, \widehat{W} , by gluing to the

$$M(K_i) = M(K_{i,1} \# \dots \# K_{i,m_i})$$

boundary component of W a copy of $V_{\#}(K_{i,1}, \dots, K_{i,m_i})$ which we call V_i . Do this for each i . The resulting 4-manifold has boundary given by $\bigsqcup_{i=1}^p \bigsqcup_{j=1}^{m_i} M(K_{i,j})$. Since the map $i_* : H_1(M(K_i)) \rightarrow H_1(V_i)$ is an isomorphism, one can use Condition (C2) to conclude that $\phi : \pi_1(W) \rightarrow \Gamma$ extends over $\pi_1(V_i)$ for each i . Specifically, if α_i is the monomorphism which exists since (W, Γ) satisfies (C2), one can define the extension of ϕ to $\pi_1(V_i)$ by the composition:

$$\pi_1(V_i) \rightarrow H_1(V_i) \xrightarrow{i_*^{-1}} H_1(M(K_i)) \xrightarrow{\alpha_i} \Gamma$$

We claim that (\widehat{W}, Γ) is a $C(k, n)$. Since the underlying group, Γ , did not change, Condition (C1) still holds. In order to see Condition (C2) consider

the following diagram:

$$\begin{array}{ccc}
 \pi_1(M(K_{i,j})) & \longrightarrow & H_1(M(K_{i,j})) \\
 \downarrow & & \downarrow \cong \\
 \pi_1(V_{\#}) & \longrightarrow & H_1(V_{\#}) \\
 \uparrow & & \downarrow i_*^{-1} \cong \\
 \pi_1(M(K_i)) & \longrightarrow & H_1(M(K_i)) \\
 \downarrow & & \downarrow \alpha_i \\
 \pi_1(W) & \longrightarrow & \Gamma
 \end{array}$$

The composition of the maps on the right hand column is the required monomorphism. Its image is contained in the Abelian subgroup, A , given by the fact that (W, Γ) satisfies (C2).

We now use Proposition 7.11 (2) to show Condition (C3). Let $\psi : \pi_1(\widehat{W}) \rightarrow \Lambda$ and S be as in the statement of Condition (C3). If x and y are in

$$P := \text{Ker} \left(\bigoplus_{i,j} H_1(M(K_{i,j}); \mathbb{Q}[\Lambda]S^{-1}) \rightarrow H_1(\widehat{W}; \mathbb{Q}[\Lambda]S^{-1}) \right)$$

then there exist x' and y' in

$$P' := \text{Ker} \left(\bigoplus_i H_1(M(K_i); \mathbb{Q}[\Lambda]S^{-1}) \rightarrow H_1(W; \mathbb{Q}[\Lambda]S^{-1}) \right)$$

such that $x - x'$ and $y - y'$ are in $Q = \bigoplus_i Q_i$ where

$$Q_i := \text{Ker} \left(H_1(\partial V_i; \mathbb{Q}[\Lambda]S^{-1}) \rightarrow H_1(V_i; \mathbb{Q}[\Lambda]S^{-1}) \right).$$

Consider the equality from the sesquilinearity of the Blanchfield form,

$$(7.5) \quad Bl_{\Lambda}(x - x', y - y') = Bl_{\Lambda}(x, y) - Bl_{\Lambda}(x, y') - Bl_{\Lambda}(x', y) + Bl_{\Lambda}(x', y').$$

Since Q is isotropic by Proposition 7.11 (2), $Bl_{\Lambda}(x - x', y - y') = 0$. By assumption, Condition (C3) holds for (W, Γ) , so P' is isotropic and $Bl_{\Lambda}(x', y') = 0$. Since x and y' are carried by different components of ∂V , as are x' and y , $Bl_{\Lambda}(x, y') = Bl_{\Lambda}(y', x) = 0$. Thus, (7.5) reduces to $0 = Bl_{\Lambda}(x, y)$ so that P is isotropic and Condition (C3) holds.

Condition (C4) holds because of Novikov additivity, since by Proposition 7.11 (1), $\sigma(V_i) = \sigma^{(2)}(V_i, \Theta) = 0$. This completes the proof. \square

Lemma 7.5. *For $i = 1, \dots, m$, let K_i be a knot, R_i be slice and η_i represent a doubly anisotropic element of $A_0(R_i)$. If $\bigsqcup_{i=1}^m M(R_i(\eta_i, K_i))$ bounds a $C(n, h)$ with $n \leq h$, (W, Γ) , then $\bigsqcup_{i=1}^m M(K_i)$ bounds a $C(n+1, h)$.*

Proof. To each $M(R_i(\eta_i, K_i))$ -boundary component of W glue a copy of $V_{\text{inf}}(R_i, K_i, \eta_i)$ which we abbreviate as V_i . Do this for each i . Call the resulting 4-manifold W_0 . It has boundary $\partial W_0 = \bigsqcup_i (M(K_i) \sqcup M(R_i))$. For each i , let E_i be the complement of a slice disk for the slice knot R_i . Let \widehat{W} be given by gluing E_i to W_0 along the $M(R_i)$ boundary component for each i .

Similar to the proof of Lemma 7.4, the coefficient system extends over $V_i \cup E$ and on $V_i \cup E$ it factors through Abelianization. Since μ_i (the meridian of K_i) is isotopic in V_i to η_i which is nullhomologous, $\phi(\mu_i)$ is trivial. Thus, $\mu_i \cong \eta_i$ lifts to a curve in the Γ -cover of \widehat{W} and can be regarded as an element of $H_1(\widehat{W}; \mathbb{Q}[\Gamma])$.

If x and y are elements of

$$P := \text{Ker} \left(\bigoplus_i H_1(M(R_i); \mathbb{Q}[\Gamma]) \rightarrow H_1(W_0; \mathbb{Q}[\Gamma]) \right)$$

then there must exist x' and y' in

$$P' := \text{Ker} \left(\bigoplus_i H_1(M(R_i(\eta_i, K_i)); \mathbb{Q}[\Gamma]) \rightarrow H_1(W; \mathbb{Q}[\Gamma]) \right)$$

such that $x - x'$ and $y - y'$ are in $S = \bigoplus_i S_i$ where

$$S_i := \text{Ker} (H_1(M(R_i); \mathbb{Q}[\Gamma]) \oplus H_1(M(R_i(\eta_i, K_i)); \mathbb{Q}[\Gamma]) \rightarrow H_1(V_i; \mathbb{Q}[\Gamma])).$$

By Proposition 7.11 (4) S is isotropic so that

$$0 = Bl(x - x', y - y') = Bl(x, y) - Bl(x, y') - Bl(x', y) + Bl(x', y').$$

By remark 7.2, P' is isotropic so that $Bl(x', y') = 0$. Since x and y' as well as x' and y sit in different components, $Bl(x, y') = Bl(x', y) = 0$. Thus, $Bl(x, y) = 0$ and P is isotropic.

Since the inclusion induced map $\frac{\pi_1(M(R_i))}{\pi_1(M(R_i))_{\text{r}}^{(1)}} \rightarrow \frac{\pi_1(M(E_i))}{\pi_1(M(E_i))_{\text{r}}^{(1)}} \cong \mathbb{Z}$ is an isomorphism, it follows that if $Q_i = \text{Ker}(A_0(R_i) \rightarrow A_0(E_i))$, then

$$\text{Ker} (H_1(M(R_i); \mathbb{Q}[\Gamma]) \rightarrow H_1(E_i; \mathbb{Q}[\Gamma])) = Q_i \otimes \mathbb{Q}[\Gamma].$$

By a Mayer-Vietoris argument,

$$\text{Ker} \left(\bigoplus_i H_1(M(R_i); \mathbb{Q}[\Gamma]) \rightarrow H_1(W; \mathbb{Q}[\Gamma]) \right) = P + \bigoplus_i (Q_i \otimes \mathbb{Q}[\Gamma]).$$

If $\langle 0, \dots, 0, \eta_j \otimes 1, 0, \dots, 0 \rangle \in \bigoplus_i H_1(M(R_i); \mathbb{Q}[\Gamma])$ were in $P + \bigoplus_i (Q_i \otimes \mathbb{Q}[\Gamma])$, then there would exist some $p = \langle p_i \rangle \in P$ and $\langle q_i \rangle \in \bigoplus_i (Q_i \otimes \mathbb{Q}[\Gamma])$ with

$\eta_j \otimes 1 = p_j + q_j$ and $0 = p_i + q_i$ when $i \neq j$. Since q_i is in the isotropic submodule $Q_i \otimes \mathbb{Q}[\Gamma]$ for each i and P is isotropic, this implies

$$\begin{aligned} 0 &= Bl_\Gamma(p, p) = \sum_i (Bl_\Gamma(p_i, p_i)) = Bl_\Gamma(p_j, p_j) + \sum_{i \neq j} Bl_\Gamma(q_i, q_i) \\ &= Bl_\Gamma(p_j, p_j) \end{aligned}$$

So that $\eta_j \otimes 1 = p_j + q_j$ sits in the sum of the isotropic submodule $Q_j \otimes \mathbb{Q}[\Gamma]$ together with the isotropic submodule generated by p_j . Corollary 6.4 then contradicts the assumption that η_j be doubly anisotropic. Thus, $\mu_j \cong \eta_j$ must be nonzero in $H_1(\widehat{W}; \mathbb{Q}[\Gamma])$ and $H_1(M(K_i))$ maps injectively to $H_1(\widehat{W}; \mathbb{Q}[\Gamma])$.

Letting $G = \text{Ker}(\pi_1(\widehat{W}) \rightarrow \Gamma)$, define $\widehat{\Gamma}$ to be the quotient $\frac{\pi_1(W)}{G_\tau^{(1)}}$ where $G_\tau^{(1)}$ is the first term in the rational derived series of G . Consider the following short exact sequence,

$$0 \rightarrow \frac{G}{G_\tau^{(1)}} \rightarrow \widehat{\Gamma} \rightarrow \Gamma \rightarrow 0.$$

It reveals first that $\widehat{\Gamma}$ is PTFA, since Γ is PTFA and $\frac{G}{G_\tau^{(1)}}$ is TFA. Secondly, since G is the fundamental group of the Γ cover of \widehat{W} , $\frac{G}{G_\tau^{(1)}}$ is the quotient of $H_1(\widehat{W}; \mathbb{Z}[\Gamma])$ by its \mathbb{Z} -torsion, into which $H_1(M(K_i))$ was shown to inject. Thus, this choice of $(\widehat{W}, \widehat{\Gamma})$ satisfies (C2). It satisfies (C1) for $n + 1$ since $\widehat{\Gamma}_\tau^{(n+1)}$ sits in the TFA group $\frac{G}{G_\tau^{(1)}}$, so that $\widehat{\Gamma}_\tau^{(k+2)} = 0$.

The argument that $(\widehat{W}, \widehat{\Gamma})$ satisfies conditions (C3) and (C4) is just as in the proof of 7.4 with part (3) of 7.11 replacing part (2). \square

REFERENCES

- [1] Evan Bullock and Christopher William Davis. Strong coprimality and strong irreducibility of Alexander polynomials. *Topology and its Applications*, 2011. to appear.
- [2] A. J. Casson and C. McA. Gordon. Cobordism of classical knots. In *À la recherche de la topologie perdue*, volume 62 of *Progr. Math.*, pages 181–199. Birkhäuser Boston, Boston, MA, 1986. With an appendix by P. M. Gilmer.
- [3] Jae Choon Cha. Amenable L^2 -theoretic methods and knot concordance. Preprint available at <http://arxiv.org/abs/1010.1058>.
- [4] Jae Choon Cha. The structure of the rational concordance group of knots. *Mem. Amer. Math. Soc.*, 189(885):x+95, 2007.
- [5] Jae Choon Cha. Topological minimal genus and L^2 -signatures. *Algebr. Geom. Topol.*, 8(2):885–909, 2008.
- [6] Tim D. Cochran. Noncommutative knot theory. *Algebr. Geom. Topol.*, 4:347–398, 2004.
- [7] Tim D. Cochran, Shelly Harvey, and Constance Leidy. Knot concordance and higher-order Blanchfield duality. *Geom. Topol.*, 13(3):1419–1482, 2009.

- [8] Tim D. Cochran, Shelly Harvey, and Constance Leidy. Derivatives of knots and second-order signatures. *Algebr. Geom. Topol.*, 10(2):739–787, 2010.
- [9] Tim D. Cochran, Shelly Harvey, and Constance Leidy. Primary decomposition and the fractal nature of knot concordance. *Math Annalen*, 2010. DOI: 10.1007/s00208-010-0604-5.
- [10] Tim D. Cochran, Kent E. Orr, and Peter Teichner. Knot concordance, Whitney towers and L^2 -signatures. *Ann. of Math. (2)*, 157(2):433–519, 2003.
- [11] Tim D. Cochran, Kent E. Orr, and Peter Teichner. Structure in the classical knot concordance group. *Comment. Math. Helv.*, 79(1):105–123, 2004.
- [12] Christopher William Davis. Von Neumann rho invariants as obstructions to torsion in the topological knot concordance group. 2010. Preprint available at <http://arxiv.org/abs/1010.5020>.
- [13] Stefan Friedl. L^2 -eta-invariants and their approximation by unitary eta-invariants. *Math. Proc. Cambridge Philos. Soc.*, 138(2):327–338, 2005.
- [14] Stefan Friedl, Constance Leidy, and Laurentiu Maxim. L^2 -Betti numbers of plane algebraic curves. *Michigan Math. J.*, 58(2):411–421, 2009.
- [15] Shelly L. Harvey. Homology cobordism invariants and the Cochran-Orr-Teichner filtration of the link concordance group. *Geom. Topol.*, 12(1):387–430, 2008.
- [16] Se-Goo Kim and Taehee Kim. Polynomial splittings of metabelian von Neumann rho-invariants of knots. *Proc. Amer. Math. Soc.*, 136(11):4079–4087, 2008.
- [17] Constance Leidy. Higher-order linking forms for 3-manifolds. Preprint.
- [18] Charles Livingston. Seifert forms and concordance. *Geom. Topol.*, 6:403–408 (electronic), 2002.
- [19] Wolfgang Lück. L^2 invariants of regular coverings of compact manifolds and CW-complexes. In *Handbook of Geometric Topology*, pages 735–817. North-Holland, Amsterdam, 2002.
- [20] Joseph J. Rotman. *Advanced modern algebra*, volume 114 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. Second edition [of MR2043445].
- [21] Bo Stenström. *Rings of quotients*. Springer-Verlag, New York, 1975. Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory.

DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY

E-mail address: `cwd1@rice.edu`